A Relation Between the Lagrangian and Eulerian Turbulent Velocity Autocorrelations

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Direct computation of the Lagrangian autocorrelation is not feasible since, generally, one cannot measure the turbulent velocity of each fluid particle. A model whereby the Lagrangian autocorrelation is determined in terms of a domain integral of a set of regular Eulerian autocorrelations is advanced. The Eulerian autocorrelations are to be acquired concurrently at all positions in the flowfield. Three novel averaging procedures are utilized for obtaining the relationship between the Lagrangian and Eulerian autocorrelations. This relationship is not constrained to either homogeneous or isotropic turbulence.

I. Introduction

DIRECT estimation of the Lagrangian autocorrelation is extremely difficult and even insurmountable due to the intrinsic problems associated with measuring the velocity of each fluid particle. Myriad attempts have been made to deduce the Lagrangian autocorrelation from the readily measurable Eulerian turbulent velocity at fixed points in space. A thorough review of these methods is not given here since the background literature is readily accessible. Their main features pertinent to the work presented herein are, however, briefly discussed. The available approaches for the estimation of the Lagrangian autocorrelation can be categorized into three broad groups based on their salient traits. These three classes are the linear-correlation, the moving-frame autocorrelation, and the probability methods. Essentially, all the available methods are based in a variety of ways on the presumption that the Lagrangian autocorrelation has to reveal similarity to some particular Eulerian correlation.

Linear-Correlation Method

The shape of the Lagrangian autocorrelation is proposed to be similar to either an axial Eulerian cross correlation[1] or a single Eulerian autocorrelation[2] in the linear-correlation approach. It is further assumed in this method that the turbulence is homogeneous and isotropic. The Lagrangian autocorrelation is then simply derived by either contracting or stretching the Eulerian correlations by means of an empirical factor. This factor does not appear to possess a unique value. The linear cross-correlation method was put forth by Mickelsen[1] in 1955, based on a mass diffusion experiment in the core of a pipe where the turbulence is isotropic. Values of the contraction factor which related the space coordinate of the Eulerian longitudinal cross correlation to the time coordinate of the Lagrangian autocorrelation varied from 0.55 to 0.725 (24% variation). The results depended upon the mean and turbulent velocity levels. An average value for this factor of roughly 0.6 was proposed for a velocity ranging from 0.55-4.27 m/s (1.8-14 ft/s).

The linear-autocorrelation approach postulates that the Lagrangian autocorrelation is identical in shape with a single Eulerian velocity autocorrelation at a fixed point in space but having a different time scale. This hypothesis was advanced by Hay and Pasquill[2] in 1959 based on the assumption that in homogeneous turbulence the Lagrangian autocorrelation decays much more slowly than the Eulerian velocity autocorrelation measured at a fixed point. The former is then obtained by linear stretching of the time coordinate of the latter. Values of the stretching factor from 1.1-8.5 (87% variation) were inferred based on a diffusion experiment at ground level. Despite this scattering, an average Lagrangian time scale of four times the Eulerian time scale was suggested. A similar linear stretching of the time coordinate in relating the Lagrangian and Eulerian autocorrelations was proposed by Angell[3] in 1964 based on monitoring the trajectories of tetroons at heights of 762 m (2500 ft). An average value of about 3.3 was put forward for the linear time coefficient. This average value was suggested in spite of observing a tendency for the time scale to increase from 1-7 (86% variation) with decreasing turbulence intensity and increasing stability. Although not explicitly stated, it appears that homogeneous turbulence was assumed. The similarity in shape of the Lagrangian autocorrelation and the Eulerian correlation measured at a fixed point (essentially the Eulerian autocorrelation) was further set forth indirectly by Snyder and Lumley[4] in 1971. In this work the motion of single spherical beads of different weights in a vertical wind tunnel in homogeneous and isotropic turbulence was monitored photographically. The heavy particle correlations were interpreted as representative of Eulerian correlations, while the light particle correlations were construed as Lagrangian autocorrelations. Based on these assumptions, the ratio of the Lagrangian to Eulerian autocorrelation integral time scales was crudely estimated to be about 3.

Moving-Frame Autocorrelation Method

In the moving-frame autocorrelation approach the Lagrangian autocorrelation is estimated by the envelope of a set of Eulerian space-time cross correlations of the longitudinal fluctuating velocity under the assumption of

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homogeneous and isotropic turbulence. This envelope, which connects the peaks of the cross correlations, is interpreted as a moving Eulerian autocorrelation which would be measured by a probe traveling steadily at the mean velocity. This scheme was put forward by Baldwin and Walsh[5] in 1961. It was further explored by Baldwin and Mickelson[6] in 1963 based on a pipe diffusion experiment in isotropic turbulence resembling that described in Ref. 1. In this method, it is implied that the moving Eulerian autocorrelation and the Lagrangian autocorrelation are of similar shape but different scales. Values of the factor relating the varying axial separation distance of the cross correlations to the time coordinate of the Lagrangian autocorrelation ranged from 1.2-0.14 (88% variation). The corresponding range of the linear autocorrelation stretching factor[2] would be 40-4.7.

A similar equivalence between the moving Eulerian time correlation and the Lagrangian autocorrelation was proposed by Deissler[7] in 1961 for decaying homogeneous turbulence. It was further shown that these two autocorrelations are approximately equal for low-turbulence levels (large decay times) and small diffusion times. Shlien and Corrsin[8] in 1974 asserted that the Lagrangian autocorrelation is rather different in shape from the moving autocorrelation based on a heat dispersion experiment in approximately isotropic turbulence. In this work, the Lagrangian autocorrelation was estimated by trial and error utilizing the Lagrangian micro and integral time scales which were computed from dispersion data.

Probability Method

The probability method is based on the conjecture that for very long time intervals the displacements of fluid particles become statistically independent of the velocity distribution in homogeneous turbulence. Then it was inferred that the Lagrangian autocorrelation can be approximated by a domain integral over the entire flowfield (volume integral) of the weighted Eulerian two-point two-time cross correlation. The weight function is the particle displacement probability density function which is generally unknown. This scheme was initially advanced by Corrsin[9] in 1959 for homogeneous turbulence and extended by Saffman[10] in 1963 for small time intervals in isotropic turbulence. In the latter case, the Lagrangian autocorrelation is expressed by an integro-differential equation for the mean-square displacement of a fluid particle in terms of the spectrum function of the Eulerian space-time cross correlation. A normal probability density function for the displacement was further assumed. The Lagrangian autocorrelation was then computed for isotropic turbulence assuming an exponential decaying spectrum. Upon comparison with the moving-frame Eulerian autocorrelation method, it was found that the ratio of the integral time scales of the Eulerian autocorrelation and the Lagrangian autocorrelation is roughly 1.25 times the longitudinal turbulence intensity for small values of the latter.[10] This result yielded a linear autocorrelation stretching factor of 5.6.

The probability approach was further explored by Kraichnan[11] in 1964 in isotropic turbulence. In this analysis, the Lagrangian and Eulerian velocity fields were expressed in terms of a passive labeling field[12] and the direct interaction approximation for this scalar field[13,14] was utilized. The Lagrangian time covariance (autocorrelation) was then represented by an average of the Eulerian space-time covariance (cross correlation) over the effective volume occupied by the particle displacement probability distribution function at any difference time (lag time). The direct-interaction approximation predicts that the Lagrangian autocorrelation falls off more rapidly than the Eulerian autocorrelation at a fixed point.[11] This leads to a linear autocorrelation factor smaller than unity, which is in conflict with its value put forward by the linear-autocorrelation method.[2]

An additional attempt to relate the Lagrangian autocorrelation to the Eulerian correlation function using the probability approach was reported by Philip[15] in 1967. In this approach, the Lagrangian autocorrelation in isotropic turbulence is estimated by a space-time integration of the weighted Eulerian correlation function of the longitudinal turbulent velocity in an arbitrary direction (space-time cross correlation). The weight function is the probability density distribution function governing the probability of finding any particle at a given position after a certain time interval. The Lagrangian autocorrelation was then expressed by an integral equation assuming a Gaussian probability density distribution function and an exponential decaying Eulerian space-time cross correlation. This integral was solved in terms of a parameter which depends upon the Eulerian axial turbulent velocity, the Eulerian integral time and the longitudinal length scales. The resulting ratio of the integral time scales of the Lagrangian autocorrelation and the moving-frame Eulerian autocorrelation increases with diminishing longitudinal turbulence intensity. This method is generally similar in many respects to the one advanced in Ref. 10. The values of the integral time scales ratio suggested in Ref. 10 are, however, two to three times larger than their counterparts deduced in Ref. 15 for the same longitudinal turbulence intensity.

The foregoing brief review of the various available methods for the estimation of the Lagrangian autocorrelation clearly reveals wide disparities in both the basic approach and useful results. In light of the significant discrepancies among these numerous attempts, further investigation for the sake of putting forth a relationship between the Lagrangian autocorrelation and Eulerian autocorrelation functions was undertaken.

II. Lagrangian Description of Fluid Motion

In the Lagrangian description, the motion of a tagged fluid particle is described in the course of time in terms of its arbitrary reference position (or initial position). The coordinates of this initial point in a fixed frame of reference, i.e., an Eulerian frame or a spatial frame, and the time are the Lagrangian independent variables.

The trajectory (or pathline) of any kth fluid particle which passes through a selected reference point A at some initial time t_A^k is denoted by $s^k[a_\ell, X_\ell(t^k)]$. Throughout this analysis, Cartesian tensor notation is utilized and, consequently, the subscripts i, j, and ℓ can take on only integer values 1, 2, or 3 unless otherwise specified. Superscripts are used for identifying the fluid particles which move past the reference point A during a certain time interval T. Pathlines of several fluid particles which cross the same reference position A are portrayed in Fig. 1. With respect to a fixed frame of reference, the coordinates of the arbitrary initial point A are $x_\ell = a_\ell$ and the kth fluid particle instantaneous position vector is designated by $X_\ell(t^k)$. Hence, the kth particle trajectory $s^k[a_\ell, X_\ell(t^k)]$ in the course of time in the spatial system of coordinates x_ℓ is described by the position vector $X_\ell(t^k)$. Each kth marked particle that passes through reference point A does so at a unique initial time t_A^k; thus, for any two particles k and n, $t_A^k \neq t_A^n$. On the other hand, the position vectors of their initial point A are exactly the same, i.e., $X_\ell(t_A^k) = X_\ell(t_A^n)$. The initial times are related by

$$t_A^n = t_A^k + (n-k)\Delta t \tag{1}$$

where $n > k$ and Δt is the smallest increment of time required for two consecutive particles to leave and arrive at reference point A. This time increment is basically unknown.

The turbulent velocity of each kth fluid element in the Lagrangian method is designated by $v_i(a_\ell, t^k)$ and is depicted along the particle pathline s^k in Fig. 1 at the reference point A and at some other point B^k. In this analysis, only the turbulent velocity is considered inasmuch as its autocorrelation is not affected by the mean velocity. It is further assumed that the turbulence is ergodic and, hence, all the turbulent statistical properties can be deduced from a single realization of the flow.[16]

III. Lagrangian Autocorrelation

The kth tagged fluid particle possesses a velocity $v_i(a_\ell,t_A^k)$ when it is at the reference point A at initial time t_A^k. At a later time, the very same kth fluid particle is at some point B^k along its trajectory s^k, as portrayed in Fig. 1. The elapsed time needed for the kth particle to travel from its initial point A to its new position B^k is τ. At this point, whose coordinates and position vectors are $x_\ell = b_\ell^k$ and $X_\ell(t_A^k+\tau)$, the particle's Lagrangian velocity is $v_i(a_\ell,t_A^k+\tau)$. The Lagrangian turbulent velocity autocorrelation is obtained by averaging the two-point velocity product (or the Lagrangian velocity product) over a large number of fluid particles $N(k=1$ to $N)$ that pass through the reference point A within a selected time interval T. This average can essentially be interpreted as a particle-space averaging with respect to these N fluid elements. The single-reference-point Lagrangian turbulent velocity autocorrelation is then given by

$$L_{ij}(a_\ell,\tau) = \frac{1}{N}\sum_{k=1}^{N} v_i(a_\ell,t_A^k)\,v_j(a_\ell,t_A^k+\tau) \tag{2}$$

The instantaneous Eulerian velocity $u_i(x_\ell,t)$ at any position along the s^k pathline is exactly equal to the Lagrangian velocity of the kth tagged fluid particle at that instant when this particle passes the very same position. Thus, instantaneously, at any point along an s^k trajectory defined by the position vector $X_\ell(t^k)$

$$u_i[X_\ell(t^k),t^k] = v_i(a_\ell,t^k) \tag{3}$$

while, when the fluid particle is at the initial position A,

$$u_i[X_\ell(t_A^k),t_A^k] = v_i(a_\ell,t_A^k) \tag{4}$$

The Eulerian autocorrelation at a fixed point in space x_ℓ of the Eulerian velocity $u_i(x_\ell,t)$ is

$$R_{ij}(x_\ell;\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T u_i(x_\ell,t)\,u_j(x_\ell,t+\tau)\,\mathrm{d}t \tag{5}$$

in which T stands for the averaging time and the semicolon on the right-hand side indicates that the Eulerian autocorrelation is computed at a specific time delay τ.

IV. Lagrangian and Eulerian Velocity Products

Basically, relating the Lagrangian and Eulerian autocorrelations consists of establishing a connection between the velocity products which comprise them. The Lagrangian velocity $v_i(a_\ell,t^k)$ of the kth tagged fluid particle that moved past the reference point A is defined at any moment only at its particular location along its trajectory $s^k[a_\ell,X_\ell(t^k)]$. On the other hand, for the same instant in time t, the Eulerian velocity $u_i(x_\ell,t)$ is basically specified at every position in space which all the fluid occupies since it is not related to any distinct fluid element. The Eulerian velocity along a pathline can be expressed as $u_i(s^k,t)$ in terms of the intrinsic coordinate s^k. The properties of the Eulerian velocity, viz., a continuous differentiable function, are preserved in both the spatial (x_ℓ) and natural (s^k) systems of coordinates.

The Eulerian velocity at any point P^k on the kth fluid element trajectory s^k can be evaluated at any instant in time by means of Taylor series expansions of its corresponding velocities at reference point A and at point B^k as illustrated in Fig. 2. In the intrinsic system of coordinates, the Eulerian velocities at points A and B^k are $u_i(0,t)$ and $u_j(s_B^k,t)$, and the coordinate of point P^k is s^k where $0<s^k<s_B^k$. The Eulerian

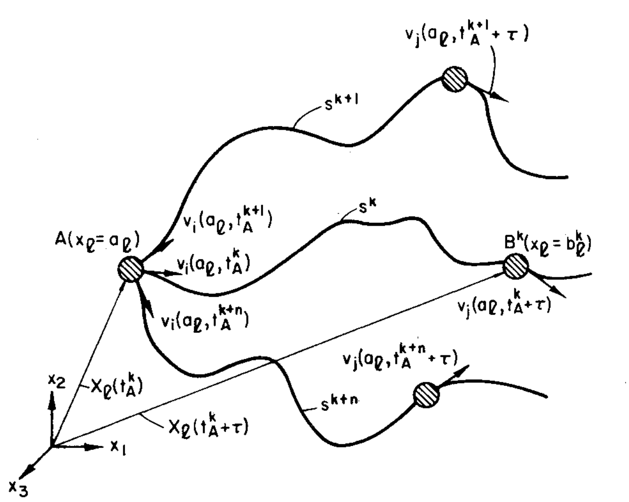

Fig. 1 Pathlines of several fluid particles which cross the same reference position A.

velocity at point P^k in terms of a Taylor series expansion about point A when the time is held at t_A^k is

$$u_i(s^k;t_A^k) = u_i(0,t_A^k) + \sum_{m=1}^{\infty}\frac{(s^k)^m}{m!}\left[\frac{\mathrm{d}^m u_i(s^k;t_A^k)}{\mathrm{d}(s^k)^m}\right]_{s^k=0} \tag{6}$$

Similarly, at time $t_B^k = t_A^k+\tau$ the Eulerian velocity at the very same point P^k estimated by a Taylor series expansion about point B^k is

$$u_j(s^k;t_A^k+\tau) = u_j(s_B^k,t_A^k+\tau) + \sum_{n=1}^{\infty}\frac{(s^k-s_B^k)^n}{n!}\left[\frac{\mathrm{d}^n u_j(s^k;t_A^k+\tau)}{\mathrm{d}(s^k)^n}\right]_{s^k=s_B^k} \tag{7}$$

In both foregoing equations the semicolon indicates that the series expansions about points A and B^k were carried out at two specific instants in time, i.e., when in each series expansion the time is held constant at t_A^k and t_B^k, respectively. The first terms in the series expansions in Eqs. (6) and (7) are exactly the particle fluid Lagrangian velocities at times t_A^k and $t_A^k+\tau$ at point P^k. This results from the instantaneous equality between the Eulerian and Lagrangian velocities in accordance with Eq. (3).

The Eulerian velocity product $r_{ij}(s^k;t_A^k,\tau)$ at any point P^k is formed by simply multiplying the velocities prevailing at this position at times t_A^k and $t_A^k+\tau$ which are given by Eqs. (6) and (7). This product yields

$$\begin{aligned} r_{ij}(s^k;t_A^k,\tau) &= u_i(s^k;t_A^k)\,u_j(s^k;t_A^k+\tau) = v_i(a_\ell,t_A^k)\,v_j(a_\ell,t_A^k+\tau) \\ &+ \sum_{m=1}^{\infty}\frac{(s^k)^m}{m!}c_{j,im}(s_B^k,0,t_A^k+\tau,t_A^k) \\ &+ \sum_{n=1}^{\infty}\frac{(s^k-s_B^k)^n}{n!}c_{i,jn}(0,s_B^k,t_A^k,t_A^k+\tau) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(s^k)^m(s^k-s_B^k)^n}{m!n!}c_{im,jn}(0,s_B^k,t_A^k,t_A^k+\tau) \end{aligned} \tag{8}$$

in which the Lagrangian velocity product is substituted for the product of the first terms in the series expansions of the Eulerian velocity in view of Eq. (3). The last three terms in Eq. (8) stand for the Eulerian velocity crossproducts, since the Eulerian two-time velocity-velocity derivative crossproducts are

$$c_{j,im}(s_B^k,0,t_A^k+\tau,t_A^k) = u_j(s_B^k,t_A^k+\epsilon)\left[\frac{\mathrm{d}^m u_i(s^k;t_A^k)}{\mathrm{d}(s^k)^m}\right]_{s^k=0} \tag{9}$$

and

$$c_{i,jn}(0,s_B^k,t_A^k,t_A^k+\tau)=u_i(0,t_A^k)\left[\frac{\mathrm{d}^n u_j(s^k;t_A^k+\tau)}{\mathrm{d}(s^k)^n}\right]_{s^k=s_B^k} \tag{10}$$

and the Eulerian two-point two-time double-velocity derivative crossproduct is

$$c_{im,jn}(0,s_B^k,t_A^k,t_A^k+\tau)=\left[\frac{\mathrm{d}^m u_i(s^k;t_A^k)}{\mathrm{d}(s^k)^m}\right]_{s^k=0}\left[\frac{\mathrm{d}^n u_j(s^k;t_A^k+\tau)}{\mathrm{d}(s^k)^n}\right]_{s^k=s_B^k} \tag{11}$$

A relationship between the Lagrangian and Eulerian velocity products of the kth fluid particle at any point P^k on its pathline is hence supplied by Eq. (8).

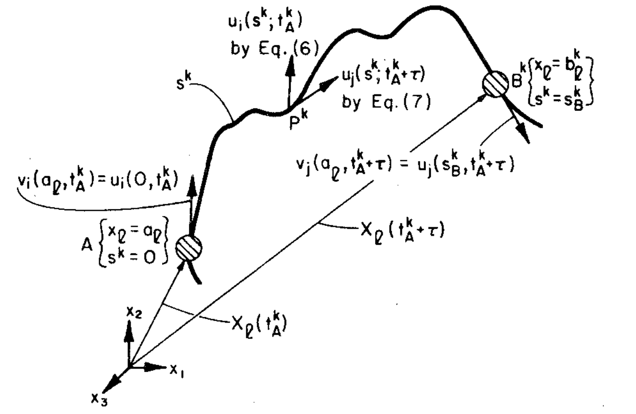

Fig. 2 Illustration of the Taylor series expansions of the Eulerian velocities at point P^k on a pathline.

V. Lagrangian-Eulerian Autocorrelation Relationship

Single-Reference-Point Lagrangian Autocorrelation

Trajectory Averaging

It is conceivable to express the Eulerian velocity product and the three Eulerian velocity crossproducts in Eq. (8) at all possible points P^k on a kth trajectory in terms of single characteristic values. These characteristic quantities are represented by their spatial mean values on the kth pathline between reference point A and position B^k. This averaging involves line integration along the kth trajectory from initial point $s^k=0$ to point s_B^k, i.e., trajectory averaging in the natural frame. Since the kth fluid particle reaches point s_B^k on its pathline after some time lapse τ the trajectory averaging is applied for each position s_B^k. The Lagrangian velocity product in Eq. (8) is not affected by this trajectory averaging inasmuch as it is independent of the intrinsic coordinate s^k. The trajectory averaging leads, after some manipulation, to the following expression for the Lagrangian velocity product[17]:

$$\begin{aligned}v_i(a_\ell,t_A^k)\,v_j(a_\ell,t_A^k+\tau)&=\frac{1}{s_B^k}\int_0^{s_B^k} r_{ij}(s^k;t_A^k,\tau)\,\mathrm{d}s^k\\&-\Bigg[\sum_{m=1}^{\infty}\frac{(s_B^k)^m}{(m+1)!}c_{j,im}(s_B^k,0,t_A^k+\tau,t_A^k)\\&+\sum_{n=1}^{\infty}\frac{(-1)^n(s_B^k)^n}{(n+1)!}c_{i,jn}(0,s_B^k,t_A^k,t_A^k+\tau)\\&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(s_B^k)^{m+n}}{(m+n+1)!}c_{im,jn}(0,s_B^k,t_A^k,t_A^k+\tau)\Bigg]\end{aligned} \tag{12}$$

It is important to point out that with regard to the trajectory averaging, the terms $c_{j,im}$, $c_{i,jn}$, and $c_{im,jn}$ are constants, since they are evaluated at fixed positions on the trajectory as indicated in Eqs. (9-11). The Lagrangian velocity product for the kth fluid particle is expressed as a result of this trajectory averaging by the spatial mean values of the Eulerian velocity product and the Eulerian velocity crossproducts along the pathline segment $s^k=0$ to s_B^k traveled by this fluid element during time τ.

Particle-Space Averaging

Throughout a time interval T, a large number of fluid particles N moved past the reference point A, as illustrated in Fig. 1. The Lagrangian velocity autocorrelation for all these N fluid elements is obtained according to Eq. (2) by particle-space averaging of their Lagrangian velocity products. Then the single-reference-point Lagrangian turbulent velocity autocorrelation is supplied by particle-space averaging of the spatial mean values of the Eulerian velocity product and Eulerian velocity crossproducts considering Eq. (12). This particle-space averaging yields unique mean values for the trajectory averages of the Eulerian velocity product and the three Eulerian velocity crossproducts for all the pathlines traced by these N fluid particles. Hence, the single-reference-point Lagrangian autocorrelation, is

$$L_{ij}(a_\ell,\tau)=\phi_{ij}^1(a_\ell,\tau)-[\phi_{ji}^2(a_\ell,\tau)+\phi_{ij}^3(a_\ell,\tau)+\phi_{ij}^4(a_\ell,\tau)] \tag{13}$$

where the elapse time (or time delay) τ can take on any value. In this equation, the first ϕ term

$$\phi_{ij}^1(a_\ell,\tau)=\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{s_B^k}\int_0^{s_B^k} r_{ij}(s^k;t_A^k,\tau)\,\mathrm{d}s^k\right] \tag{14}$$

is the particle-space average of the spatial mean value (or trajectory average) of the Eulerian velocity product, and the other three ϕ terms

$$\phi_{ji}^2(a_\ell,\tau)=\frac{1}{N}\sum_{k=1}^{N}\left[\sum_{m=1}^{\infty}\frac{(s_B^k)^m}{(m+1)!}c_{j,im}(s_B^k,0,t_A^k+\tau,t_A^k)\right] \tag{15}$$

$$\phi_{ij}^3(a_\ell,\tau)=\frac{1}{N}\sum_{k=1}^{N}\left[\sum_{n=1}^{\infty}\frac{(-1)^n(s_B^k)^n}{(n+1)!}c_{i,jn}(0,s_B^k,t_A^k,t_A^k+\tau)\right] \tag{16}$$

and

$$\phi_{ij}^4(a_\ell,\tau)=\frac{1}{N}\sum_{k=1}^{N}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(s_B^k)^{n+m}}{(m+n+1)!}c_{im,jn}(0,s_B^k,t_A^k,t_A^k+\tau)\right] \tag{17}$$

are the particle-space averages of the spatial mean values (trajectory averages) of the Eulerian velocity crossproducts. In Eqs. (14-17), the terms in the brackets are the trajectory averages according to Eq. (12), whereas the summation over k designates the particle-space averaging.

Notion of a Turbulence "Box"

The single-reference-point Lagrangian autocorrelation given by Eq. (13) expresses the average characteristics of the turbulence along the fluid particles trajectories which originated at the selected reference point A. For any flow situation it is possible to generally choose innumerable reference points. It is conceivable, however, to restrict the selection of those points within a particular finite plane. The most convenient and practical choice is apparently a plane normal to the main flow direction. Then the intrinsic nature of all the single-reference-point Lagrangian autocorrelations defined with respect to all possible reference points in this

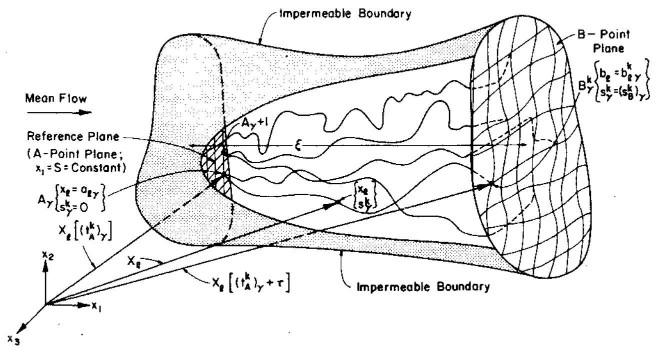

Fig. 3 Illustration of a hypothetical turbulence "box."

plane is the specification of the downstream average turbulence properties. This control surface, which is the locus of all the A-reference points, can thus be viewed as a reference plane or an A-point plane. In either confined and/or unconfined flows, such a reference plane can be readily envisioned. For wind tunnel and/or pipe flows this reference plane is basically their cross sections at any desired streamwise position. All fluid particles must move past such a plane. In the case of atmospheric and/or wake flows, it is always feasible and, moreover, desirable to adequately delineate the flow region of interest by means of suitable imaginary boundaries.

A finite reference plane, which includes all the Γ relevant A-reference points, viz., A_γ where $\gamma = 1$ to Γ, can thus be introduced for any flow situation. The picture of the pathlines traced by the N fluid particles that pass through a single reference point A portrayed in Fig. 1 applies to all the A_γ-reference points in the reference plane. It can be theorized further that all the $\Gamma \times N$ trajectories, which arise from the N fluid particles that pass each of the Γ reference points in the A-point plane during a time interval T, describe a continuous expanding or contracting control volume with increasing time lapse τ. All the pathlines that originated within the reference plane are confined within this control volume. Each of the $\Gamma \times N$ fluid particles crosses its particular A_γ-reference point at some initial time $(t_A^k)_\gamma$ and, in moving along its trajectory, reaches some new position B_γ^k after a time lapse τ. Thus, it is possible to consider a bounded control volume which encloses all the $\Gamma \times N$ trajectory segments traced throughout the time displacement τ. The locus of all B_γ^k points which contains all the locations of the $\Gamma \times N$ fluid particles at times $(t_A^k)_\gamma + \tau$ is the B-point plane. This finite control volume is thus demarcated by the A- and B-point planes. Apparently, this control volume can be interpreted as a turbulence "box." In the limiting case when $\tau = 0$, the turbulence "box" reduces to a turbulence "plane" which is the reference plane (the A-point plane) since the B-point plane collapses on it. The turbulence box becomes, moreover, the entire flowfield downstream of the reference plane, i.e., a flowfield extending to infinity, as $\tau \to \infty$.

The turbulence box can be visualized by the following hypothetical flow situation: 1) mean flow only in the x_1 direction; 2) an arbitrary control surface, i.e., reference plane, normal to the x_1 axis which is defined by $x_1 = S = \text{const}$; 3) all $\Gamma \times N$ fluid particles entering the control volume V, i.e., the turbulence box, pass through this A-point plane; 4) the envelope of the control volume (all lateral surfaces) are impermeable; and 5) all $\Gamma \times N$ fluid particles arrive after some time lapse τ at a plane B, i.e., the B-point plane. In general, the turbulence box is not predetermined since its volume is, in fact, a function of the time lapse τ. Furthermore, the B-point plane is a distorted and complex surface. This hypothetical flow situation is illustrated in Fig. 3. A cut through the turbulence box is also shown in Fig. 3 to provide a better view of the A-point plane and several trajectories. The separation ξ between the reference and B-point planes, which is shown in this figure, can be approximated using a characteristic mean velocity scale U_c and the time lapse τ according to

$$\xi = U_c \tau \tag{18}$$

It is of utmost importance to point out that the mean velocity scale U_c is representative of either a uniform mean flow or a shear mean flow according to the flow situation being considered. Practically, this streamwise length ξ can be viewed as representative of the final position $(s_B^k)_\gamma$ for all the $\Gamma \times N$ fluid particles. This longitudinal extent ξ can be furthermore interpreted as a turbulence "line" within the turbulence box.

Reference-Plane Average Lagrangian Autocorrelation

Reference-Plane Averaging

The problem of interest is to estimate the overall properties of the turbulence within the turbulence box. A plausible description of these properties can be supplied by the average of all single-reference-point Lagrangian autocorrelations over all A_γ-reference points in the reference plane S. To this end, the coordinates of the single reference point a_ℓ, the natural coordinate of any position s^k along the trajectories originated at this point and the initial time t_A^k are superseded in Eqs. (13-17) by $a_{\ell\gamma}$, s_γ^k, and $(t_A^k)_\gamma$, respectively. This replacement accounts for the A_γ-reference points. The average of the single-reference-point Lagrangian autocorrelations $L_{ij}(a_{\ell\gamma},\tau)$ with respect to all corresponding A_γ-reference points, i.e., reference-plane averaging, is simply achieved by summing Eq. (13) over $\gamma = 1$ to Γ. This reference-plane average Lagrangian autocorrelation is therefore expressed by

$$L_{ij}(S,\tau) = \frac{1}{\Gamma} \sum_{\gamma=1}^{\Gamma} L_{ij}(a_{\ell\gamma},\tau) \tag{19}$$

where S is the A-point plane or the reference plane. Next, the reference-plane Lagrangian autocorrelation is expressed in terms of the representative Eulerian quantities for the turbulence box by substituting Eq. (13) into Eq. (19). This substitution yields

$$L_{ij}(S,\tau) = \Psi_{ij}^1(S,\tau) - [\Psi_{ji}^2(S,\tau) + \Psi_{ij}^3(S,\tau) + \Psi_{ij}^4(S,\tau)] \tag{20}$$

where the reference-plane average of the Eulerian velocity product is

$$\Psi_{ij}^1(S,\tau) = \frac{1}{\Gamma} \sum_{\gamma=1}^{\Gamma} \phi_{ij}^1(a_{\ell\gamma},\tau) \tag{21}$$

and the reference-plane averages of the three Eulerian velocity crossproducts are

$$\Psi_{ji}^2(S,\tau) = \frac{1}{\Gamma} \sum_{\gamma=1}^{\Gamma} \phi_{ji}^2(a_{\ell\gamma},\tau) \tag{22}$$

$$\Psi_{ij}^3(S,\tau) = \frac{1}{\Gamma} \sum_{\gamma=1}^{\Gamma} \phi_{ij}^3(a_{\ell\gamma},\tau) \tag{23}$$

and

$$\Psi_{ij}^4(S,\tau) = \frac{1}{\Gamma} \sum_{\gamma=1}^{\Gamma} \phi_{ij}^4(a_{\ell\gamma},\tau) \tag{24}$$

in which Eqs. (14-17) are written in terms of $a_{\ell\gamma}$, s_γ^k, and $(t_A^k)_\gamma$.

The foregoing four Ψ terms are essentially the characteristic values of the Eulerian velocity product and the Eulerian velocity crossproducts for the entire turbulence box. They resulted from trajectory, particle-space, and, finally, reference-plane averaging procedures. Basically, these three averaging processes represent an interweaving of space and time averagings within the turbulence box. These intrinsic features lead naturally to the inference that these Ψ terms can be estimated by ordinary space-time averaging. The space under consideration is the volume V of the turbulence box, whereas the time during which these averages are undertaken is the time interval T necessary for the $\Gamma \times N$ fluid particles to pass through the reference plane.

The elements constituting the $\Psi^1_{ij}(S,\tau)$ term, which is given by Eq. (21), are the Eulerian velocity products $r_{ij}[s^k_\gamma;\ (t^k_A)_\gamma,\tau]$ at all points within the turbulence box. These products can be arranged in time sequences at every point x_ℓ (or s^k_γ) in this box. Such a typical position is shown in Fig. 3. Due to this grouping the Eulerian velocity product at each point x_ℓ can be expressed for all time t by $r_{ij}(x_\ell,t,\tau)$. Each time sequence comprises all successive velocity products during the time interval T at any point. Subsequent time averaging of such a time sequence furnishes exactly the local ordinary Eulerian autocorrelation

$$R_{ij}(x_\ell;\tau) = \frac{1}{T}\int_0^T r_{ij}(x_\ell;t,\tau)\,\mathrm{d}t \tag{25}$$

at position x_ℓ at a specific time delay τ. Then the domain integral over the volume V of the turbulence box of all the common Eulerian autocorrelations supplies the equivalent space-time average representation of the reference-plane average value of $\Psi^1_{ij}(S,\tau)$, viz.,

$$\Psi^1_{ij}(S,\tau) = \frac{1}{V}\int_V R_{ij}(x_\ell;\tau)\,\mathrm{d}V \tag{26}$$

The constituents of the remaining three Eulerian Ψ terms in Eq. (20) consist of velocity-velocity derivative [Eqs. (22) and (23)] and double-velocity derivative [Eq. (24)] crossproducts at positions located solely in the A- and B-point planes along the same trajectory s^k_γ. The coordinates of such a pair of points are $x_\ell = a_{\ell\gamma}$ and $b_\ell = b^k_{\ell\gamma}$ or $s^k_\gamma = 0$, and $(s^k_B)_\gamma$ in the spatial and natural frames, respectively. These coordinates of such a pair of points are portrayed in Fig. 3. In a similar manner, as for $\Psi^1_{ij}(S,\tau)$, the crossproducts composing these three Ψ terms can be rearranged into time sequences for every pair of points. As a result of this sorting the velocity-velocity derivative crossproducts can be represented at all time t by $c_{j,im}(b_\ell,x_\ell,t+\tau,t)$ and $c_{i,jn}(x_\ell,b_\ell,t,t+\tau)$. Similarly, the double-velocity derivative crossproduct can be expressed at all time t by $c_{im,jn}(x_\ell,b_\ell,t,t+\tau)$. Time averaging over time sequences of the foregoing crossproducts during time interval T yields the Eulerian space-time crosscorrelation functions

$$C_{j,im}(b_\ell,x_\ell,\tau) = \frac{1}{T}\int_0^T c_{j,im}(b_\ell,x_\ell,t+\tau,t)\,\mathrm{d}t \tag{27}$$

$$C_{i,jn}(x_\ell,b_\ell,\tau) = \frac{1}{T}\int_0^T c_{i,jn}(x_\ell,b_\ell,t,t+\tau)\,\mathrm{d}t \tag{28}$$

and

$$C_{im,jn}(x_\ell,b_\ell,\tau) = \frac{1}{T}\int_0^T c_{im,jn}(x_\ell,b_\ell,t,t+\tau)\,\mathrm{d}t \tag{29}$$

The space-time average values for Ψ^2, Ψ^3, and Ψ^4 terms are estimated by simple area integration over the reference plane S of the preceding three space-time cross correlations. Hence, the equivalent space-time average representations of the reference-plane averages of the last three Ψ terms in Eq. (20) are:

$$\Psi^2_{ji}(S,\tau) = \sum_{m=1}^{\infty}\frac{(U_c\tau)^m}{(m+1)!}\,\frac{1}{S}\int_S C_{j,im}(b_\ell,x_\ell,\tau)\,\mathrm{d}S \tag{30}$$

$$\Psi^3_{ij}(S,\tau) = \sum_{n=1}^{\infty}\frac{(-1)^n(U_c\tau)^n}{(n+1)!}\,\frac{1}{S}\int_S C_{i,jn}(x_\ell,b_\ell,\tau)\,\mathrm{d}S \tag{31}$$

and

$$\Psi^4_{ij}(S,\tau) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(U_c\tau)^{m+n}}{(m+n+1)!}\,\frac{1}{S}\int_S C_{im,jn}(x_\ell,b_\ell,\tau)\,\mathrm{d}S \tag{32}$$

In Eqs. (30-32) the distance along each trajectory $(s^k_B)_\gamma$ was approximated by the length ξ of the turbulence line in accordance with Eq. (18). Computation of all four Eulerian Ψ terms is evidently contingent upon simultaneous knowledge of the Eulerian autocorrelations $R_{ij}(x_\ell;\tau)$ at all points in the turbulence box and of the Eulerian cross correlations $C_{j,im}(b_\ell,x_\ell,\tau)$, $C_{i,jn}(x_\ell,b_\ell,\tau)$ and $C_{im,jn}(x_\ell,b_\ell,\tau)$ at all positions in the A- and B-point planes. This involves, in practice, the use of an array of probes which monitor concurrently the turbulent velocity at all points of interest within a flowfield.

Simplified Relationship

It is conceivable to disregard the last three Ψ terms in Eq. (20) since they are composed of cross correlations between velocity and/or velocity derivatives. The interrelation between the turbulent velocity and its derivatives is progressively weakened with increasing order of differentiation. As a result, correlations between a turbulent velocity and the derivative of another turbulent velocity or between two turbulent velocity derivatives cannot be expected to take on values even as large as the correlation between the actual velocities. With augmenting streamwise distance ξ of the turbulence box, and hence, with time lapse τ, the correlation between two velocities in the A- and B-point planes generally decreases. Then in all likelihood, the three Eulerian space-time cross correlations vanish rapidly with increasing space and time separations. Therefore, the Lagrangian autocorrelation can be practically approximated in terms of the usual Eulerian autocorrelations by the relationship

$$L_{ij}(S,\tau) \simeq \frac{1}{V}\int_V R_{ij}(x_\ell;\tau)\,\mathrm{d}V \tag{33}$$

whenever the last three Eulerian Ψ terms, i.e., the Eulerian velocity crossproducts, can be neglected. Experimental examination of the Eulerian space-time cross correlations given by Eqs. (27-29) is, however, imperative to ascertain as to whether these terms can safely be neglected.

In obtaining the foregoing Lagrangian-Eulerian autocorrelation relationship, no restrictions concerning the nature of turbulence were put forth. Consequently, this relationship for the Lagrangian autocorrelation [Eq. (33)] is not constrained to either homogeneous and/or isotropic turbulence. Such ideal flows, on the other hand, enable considerable simplification of the former expression. If the flow is homogeneous the Eulerian autocorrelation is independent of its position x_ℓ within the turbulence box, i.e., $R_{ij}(x_\ell,\tau) = R_{ij}(\tau)$. Then, it follows formally from Eq. (33), that

$$L_{ij}(S,\tau) = L_{ij}(\tau) \simeq R_{ij}(\tau) \tag{34}$$

In isotropic turbulence the Eulerian autocorrelation is moreover independent of direction, i.e., $R_{ij}(\tau) = R(\tau)$. The corresponding Lagrangian autocorrelation can then be deduced from a single Eulerian autocorrelation since Eq. (34) becomes

$$L_{ij}(\tau) = L(\tau) \simeq R(\tau) \tag{35}$$

Truly homogeneous and/or isotropic turbulence are not realizable, particularly in the atmosphere. Hence, Eq. (33) is to be utilized for estimation of the Lagrangian autocorrelation. It is important to underscore that in order to estimate the Lagrangian autocorrelation by Eq. (33), it is vital to simultaneously secure the Eulerian autocorrelations $R_{ij}(x_i,\tau)$ at all points within the turbulence box.

It is interesting to mention that the Lagrangian autocorrelation expression for homogeneous turbulence given by Eq. (34) is intrinsically similar to the linear-autocorrelation model set forth by Hay and Pasquill[2] for the particular case when the stretching factor equals one. In addition, the equality of the Lagrangian and Eulerian autocorrelations in isotropic turbulence with zero mean flow was suggested by Deissler[7] in the moving-frame autocorrelation approach. The concept of a turbulence box was indirectly advanced in the probability method inasmuch as a domain integral was employed in estimating the Lagrangian autocorrelation.

VI. Conclusions

A relationship between the Lagrangian and Eulerian autocorrelations was developed based on three novel trajectory, particle-space, and reference-plane averagings of Eulerian velocity products. The Lagrangian autocorrelation is expressed in this model by a domain integral over a set of ordinary Eulerian autocorrelations which are to be obtained concurrently at all positions in the flowfield of interest. Such a flowfield is viewed as a turbulence "box." The relationship for the Lagrangian autocorrelation is not constrained to either homogeneous and/or isotropic turbulence.

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